Polyhedral Products

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1 Polyhedral Products

Definition 1.1. *Let $\mathcal{K}$ be a simplicial complex on $[m]$ and*

$$(\mathbf{X}, \mathbf{A}) = \{(X_1, A_1), \ldots, (X_m, A_m)\}$$

be a collection of m pairs of spaces, $A_i \subset X_i$. For each simplex $I \in [m]$ we set

$$(\mathbf{X}, \mathbf{A})^I = \{(x_1, \ldots, x_m) \in \prod X_j : x_j \in A_j \text{ for } j \notin I\}$$

and define the polyhedral product of $((X), \mathbf{A})$ corresponding $\mathcal{K}$ by

$$(\mathbf{X}, \mathbf{A})^{\mathcal{K}} = \bigcup (\mathbf{X}, \mathbf{A})^I$$

One can also consider the category obtained from the complex $\mathcal{K}$, cat($\mathcal{K}$) and take the limit

$$(\mathbf{X}, \mathbf{A})^{\mathcal{K}} = \mathrm{colim}_{\mathrm{I} \in \mathcal{K}} (\mathbf{X}, \mathbf{A})^{\mathrm{I}}$$

Definition 1.2. *We call (X, A) a monoid pair of spaces if X is endowed with an associative multiplication and unit which are both continuous maps and A is a submonoid. A map of pairs of monoids is a map of pairs of spaces which is also a monoid morphism.*

Given a monoid pair (X, A), one can define the action of A on X by left multiplication. So given a product of monoids $\prod X_i$, and submonoids $\prod A_i$, one can define an action of this product by coordinate wise left multiplication.

If (X, A) is a monoid pair and $\phi : [l] \to [m]$ is a set map, then it induces a map

$$\psi : X^l \to X^m \ (x_1, \ldots, x_l) \mapsto (y_1, \ldots, y_m), \tag{1}$$

where

$$y_j = \prod_{i \in \phi^- 1(j)} x_i.$$

Proposition 1.1. *If (X, A) is a monoid pair, then $(X, A)^{\mathcal{K}}$ is an invariant subspace of $\prod X_i$ with respect to the action of $\prod A_i$ on $\prod X_i$.*

Proof. $(X, A)^I$ is clearly invariant for $I \in \mathcal{K}$. □

Proposition 1.2. *(a) A set of map of pairs $(X, A) \to (X', A')$ induces a map of polyhedral products $(X, A)^{\mathcal{K}} \to (X', A')^{\mathcal{K}}$. If two sets of maps component wise homotopic, then the induced maps are also homotopic.*

(b) An inclusion of simplicial subcomplex $\mathcal{L} \hookrightarrow \mathcal{K}$ induces an inclusion of polyhedral products $(X, A)^{\mathcal{L}} \hookrightarrow (X, A)^{\mathcal{K}}$.

(c) If (X, A) is a monoid pair, then for any simplicial map $\phi : \mathcal{L} \to \mathcal{K}$ of simplicial complexes on sets $[l]$ and $[m]$ respectively, the map (1) *restricts to a map of polyhedral products $\psi : (X, A)^{\mathcal{L}} \to (X, A)^{\mathcal{K}}$.*

(d) If (X, A) is a commutative monoid pair, then the restriction

$$\psi|_A : A^l \to A^m$$

of (1) *is a homomorphism, and the induced map in the polyhedral products is weakly equivariant.*

Proof. (a) To see that there is an induced map of polyhedral products, notice that there is an induced map for each simplex $I \in \mathcal{K}$ and that these maps induced for $I, J \in \mathcal{K}$ are compatible on the intersections. Given a component wise homotopy from (X, A) to (X', A'), i.e. a map $(X \times \mathbb{I}, A \times \mathbb{I})$ to (X', A'), there is an induced map of polyhedral products

$$(X \times \mathbb{I}, A \times \mathbb{I})^{\mathcal{K}} \to (X', A')^{\mathcal{K}}$$

where $(X \times \mathbb{I}, A \times I)^{\mathcal{K}} \equiv (X, A)^{\mathcal{K}} \times \mathbb{I}^m$. Restricting along the diagonal on $\mathbb{I}^m$ gives the desired homotopy.

(b) Note that $\mathcal{I} \in \mathcal{L}$ implies $\mathcal{I} \in \mathcal{K}$.

(c) Note that for any $\mathcal{I} \subset [m]$, $\psi((X, A)^{\mathcal{I}}) \subset (X, A)^{\phi(\mathcal{I})}$

(d) Direct computation using equation (1).

□

For our use case, the example of most value is when $X = D^2$ and $A = S^1$. We define a moment angle complex on $\mathcal{K}$ as the polyhedral product $\mathcal{Z}_{\mathcal{K}} := (D^2, S^1)^{\mathcal{K}}$

Note that there is a natural action of $T^m = (S^1)^m$ on $(D^2, S^1)^{[m]}$ and $\mathcal{Z}_{\mathcal{K}}$ is an invariant subspace. This is going to be the main object of study for us. But before we can start deriving other results, we need to discuss some algebra associated to a simplicial complex.

2 Face Ring of a Complex

Definition 2.1. *The face ring of a simplicial complex $\mathcal{K}$ on the set $[m]$ is the quotient of the graded ring*

$$k[\mathcal{K}] = k[v_1, \ldots, v_m]/\mathcal{I}_{\mathcal{K}}$$

where $\mathcal{I}_{\mathcal{K}}$ is the ideal generated by all monomials of the form $\prod_{i \in I} v_i$ for $I \notin \mathcal{K}$.

These are also called Stanley Reisner rings. We will only need the following one result about it.

Proposition 2.1. *The face ring $k[\mathcal{K}]$ has the k vector space basis consisting of all the monomials $v_{j_1}^{a_1} \cdots v_{j_k}^{a_k}$, such that $a_i > 0$ and $\{j_1, \ldots, j_k\} \in \mathcal{K}$.*

3 Principal Bundles and Borel Construction

Let X be a Hausdorff space and G a Hausdorff topological group. G acts on X if for each $g \in G$, there is a homeomorphsim $\phi_g : X \to X$ that respects the algebraic and topological structure. A continuous map $f : X \to Y$ of spaces equipped with G actions is called equivariant if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$. It is weakly equivariant if given $\psi : G \to G$ an automorphism, $f(g \cdot x) = \psi(g) \cdot f(x)$.

A principal G bundle is a locally trivial bundle $p : E \to B$ such that G acts on E preserving fibres and the induced G action on each fiber is free and transitive. Notice that this implies that all the fibers are homeomorphic to G.

If G is a compact Lie group, then there exists a principal G bundle $EG \to BG$ whose total space EG is contractible. If $E \to B$ is another principle G bundle then there is a unique upto homotopy map $f : B \to BG$ such that E is the pullback of EG along f. EG is called the universal G space and BG the classifying space for free G actions.

If X is any G space, the diagonal action on $EG \times X$ given by $g \cdot (e, x) = (g \cdot e, g \cdot x)$ is free. The orbit space is denoted by $EG \times_G X$. There are two projections associated to this space, one induced by the projection $EG \times X \to X$ given by $EG \times_G X \to X/G$ and the other induced by $EG \times X \to EG$ given by $EG \times_G X \to BG$. Since the action of G on EG is free, the fiber for the second one is X.

In fact, if G is compact Hausdorff, EG is contractible and so $EG \times_G X$ is homotopy equivalent to X/G when the G action is free. This bundle $EG \times_G X \to BG$ with fibre X is called the bundle associated with the G space X. The cohomology $H^*(EG \times_G X; k)$ is called the equivariant cohomology $H^*_G(X, k)$ of the G space X.

4 Equivariant cohomology of $\mathcal{Z}_{\mathcal{K}}$

Specifically note that $BT^m = (\mathbb{C}P^\infty)^m$ and $ET^m = (S^\infty)^m$. The integral cohomology ring of $(\mathbb{C}P^\infty)^m$ is $\mathbb{Z}[v_1, \ldots, v_m]$, $|v_i| = 2$. $\mathbb{C}P^\infty$ has a cell decomposition with one cell in every even dimension, so $(\mathbb{C}P^\infty)^m$ has a canonical cell decomposition induced by the product. $(\mathbb{C}P^\infty)^{\mathcal{K}}$ is a cellular subcomplex of $(\mathbb{C}P^\infty)^m$ and so it also has no cells in odd dimensions. This allows us to come to the following proposition

Proposition 4.1. *The cohomology ring of $(\mathbb{C}P^\infty)^{\mathcal{K}}$ is isomorphic to the face* *ring $\mathbb{Z}[\mathcal{K}]$. The inclusion $(\mathbb{C}P^\infty)^\mathcal{K} \hookrightarrow (\mathbb{C}P^\infty)^m$ induces the quotient map*

$$\mathbb{Z}[v_1, \ldots, v_m] \to \mathbb{Z}[v_1, \ldots, v_m]/\mathcal{I}_\mathcal{K} = \mathbb{Z}[\mathcal{K}].$$

Proof. Since both $(\mathbb{C}P^\infty)^\mathcal{K}$ and $(\mathbb{C}P^\infty)^m$ only have cells in even dimensions, the cohomology of both spaces coincides with their cellular cochain complexes. Let D_j^{2k} denote the $2k$ dimensional cell in the j^{th} component of $(\mathbb{C}P^\infty)^m$. The cellular cochain group has basis of cochains $(D_{j_1}^{2k_1} \cdots D_{j_p}^{2k_p})^*$ dual to the product cells $D_{j_1}^{2k_1} \times \cdots \times D_{j_p}^{2k_p}$. The cochain map induced by the inclusion $(\mathbb{C}P^\infty)^\mathcal{K} \hookrightarrow (\mathbb{C}P^\infty)^m$ is an epimorphism with kernel generated by those chains $(D_{j_1}^{k_1} \cdots D_{j_p}^{k_p})^*$ such that $\{j_1, \ldots, j_p\} \notin \mathcal{K}$. Under the identification of $C^*((\mathbb{C}P^\infty)^m)$ with $\mathbb{Z}[v_1, \ldots, v_m]$, a cochain is mapped to the monomial $v_{j_1}^{k_1} \cdots v_{j_p}^{k_p}$. Using Proposition 2.1, we can now conclude the result. $\square$

The moment angle complex over $\mathcal{K}$, $\mathcal{Z}_\mathcal{K}$ inherits an action of T^m from it's action on $(D^2)^m$. So we can consider the Borel construction $ET^m \times_{T^m} \mathcal{Z}_\mathcal{K}$ for the T^m space $\mathcal{Z}_\mathcal{K}$. As it turns out, this is space homotopy equivalent to $(\mathbb{C}P^\infty)^\mathcal{K}$ and we have the following factoring of the inclusion map.

Proposition 4.2. *The inclusion $i : (\mathbb{C}P^\infty)^\mathcal{K} \hookrightarrow (\mathbb{C}P^\infty)^m$ factors into a composition of homotopy equivalence*

$$h : (\mathbb{C}P^\infty)^\mathcal{K} \to ET^m \times_{T^m} \mathcal{Z}_\mathcal{K}$$

and the fibration $p : ET^m \times_{T^m} \mathcal{Z}_\mathcal{K} \to BT^m = (\mathbb{C}P^\infty)^m$ with fibre $\mathcal{Z}_\mathcal{K}$.

In particular $\mathcal{Z}_\mathcal{K}$ is the homotopy fibre of the canonical inclusion i.

Proof. We note that $\mathcal{Z}_\mathcal{K} = \bigcup_\mathcal{I} (D^2, S^1)^\mathcal{I}$ and so we have the decomposition

$$\begin{aligned} ET^m \times_{T^m} \mathcal{Z}_\mathcal{K} =& \bigcup_\mathcal{I} (ET^m \times_{T^m} (D^2, S^1)) \\ =& \bigcup_I (S^\infty \times_{S^1}, S^\infty \times_{S^1} S^1)^\mathcal{I} \\ =& (S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1)^\mathcal{K} \end{aligned}$$

Now consider the diagram

$$\begin{tikzcd} pt \arrow[r] \arrow[d] & S^\infty \times_{S^1} S^1 \arrow[r] \arrow[d] & pt \arrow[d] \\ \mathbb{C}P^\infty \arrow[r, "j"] & S^\infty \times_{S^1} D^2 \arrow[r, "f"] & \mathbb{C}P^\infty \end{tikzcd}$$

where j is the inclusion of the zero section in a disc bundle and $\mathbb{C}P^\infty = S^\infty/S^1$ and f is the projection

$$f : S^\infty \times_{S^1} D^2 \to (S^\infty \times_{S^1} D^2)/(S^\infty \times_{S^1} S^1) \cong \mathbb{C}P^\infty$$

modding out the complement of the zero section. Since $S^\infty \times_{S^1} S^1 = S^\infty$ and D^2 are contractible, the composite maps $f \circ j$ and $j \circ f$ are homotopic to the identity map. Therefore we have the homotopy equivalence of pairs $(\mathbb{C}P^\infty, pt) \to (S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1)$, which induces a homotopy equivalence of polyhedral products by Proposition 1.2. Now to factor the inclusion, consider the following diagram

$$\begin{tikzcd} pt \arrow[r] \arrow[d] & S^\infty \times_{S^1} S^1 \arrow[r] \arrow[d] & \mathbb{C}P^\infty \arrow[d, equal] \\ \mathbb{C}P^\infty \arrow[r, "j"] & S^\infty \times_{S^1} D^2 \arrow[r, "g"] & \mathbb{C}P^\infty \end{tikzcd}$$

where now g is projection of the disc bundle onto the base. By passing to the induced maps of polyhedral products we obtain

$$i : (\mathbb{C}P^\infty)^{\mathcal{K}} \to (S^\infty \times_{S^1} D^2, S^\infty \times_{S^1} S^1)^{\mathcal{K}} \to (\mathbb{C}P^\infty, \mathbb{C}P^\infty)^{\mathcal{K}}$$

$\square$

Corollary 4.3.

$$H^*_{T^m}(\mathcal{Z}_{\mathcal{K}}) \cong \mathbb{Z}[\mathcal{K}]$$